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On space–time carrying a total hyperbolic skew symmetric Killing vector field

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Abstract

The complex vectorial formalism on a general space–time (M, g) was constructed by Cahen, Debever and Defrise. This formalism is based on the local isomorphism $I : \mathcal{L}(4) \to SO^3(\mathbb{C})$, where $\mathcal{L}(4)$ is the four-dimensional Lorentz group acting on the tangent spaces T_pM and $SO^3(\mathbb{C})$ is the three-dimensional complex rotation group. In this framework, the congruence of Debever plays a distinguished role. Its properties determine the general space–time M, in terms of Petrov's classification.

In the present paper, we assume that any hyperbolic vector field X on M is a skew symmetric Killing vector field having a spatial vector field Y as generative. The existence of such a vector field X is determined by an exterior differential system in involution. It is shown that M is the local Riemannian product $M = M_h \times M_s$, where M_h (resp. M_s) is a totally geodesic and totally pseudo-isotropic hyperbolic (resp. spatial) surface (the Gauss map is ametric). Any such M is a space–time of type D in Petrov's classification.

It is proved that the congruence of Debever is of electric type; in particular, it is geodesic and shear 1-free. Other geometric properties on such a general space–time are obtained.

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1. Preliminaries

Let (M, g) be a general space–time with metric tensor g. In the following, we shall make use of the complex vectorial formalism (CVF) constructed by Cahen et al. [1].

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This formalism is based on the local isomorphism $I : \mathcal{L}(4) \to SO^3(\mathbb{C})$, where $\mathcal{L}(4)$ is the four-dimensional Lorentz group acting on the tangent spaces T_pM of an orientable space–time (M, g) and $SO^3(\mathbb{C})$ is the three-dimensional complex rotation group.

Let $S = \{h_A; A \in \{1, 2, 3, 4\}\}$ be a Sachs (or a null) frame over M and $\{\theta^A\}$ its dual coframe. The vector fields h_A of S satisfy

$$g(h_1, h_4) = 1, \qquad g(h_2, h_3) = -1$$

and all the other products are 0 (h_1, h_4 are real null vectors, whilst h_2, h_3 are complex conjugates).

The six-dimensional space $\mathcal{L}^* \wedge (2)$ of 2-forms $\theta^A \wedge \theta^B$ is isomorphic to the space spanned by the 2-forms Z^α ($\alpha = 1, 2, 3$), which together with their complex conjugate $\bar{Z}^\alpha$ form a basis of the complex space $\mathbb{C}^3$. This isomorphism is defined by

$$Z^1 = \theta^3 \wedge \theta^4, \qquad Z^2 = \theta^1 \wedge \theta^2, \qquad Z^3 = \tfrac{1}{2}(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3) \tag{1.1}$$

and their corresponding complex conjugate

$$\bar{Z}^1 = \theta^2 \wedge \theta^4, \qquad \bar{Z}^2 = \theta^1 \wedge \theta^3, \qquad \bar{Z}^3 = \tfrac{1}{2}(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3). \tag{1.2}$$

With these 2-forms, the connection forms ω^A_B corresponding to $\{h_A\}$ may be expressed by the spinorial coefficients σ_α of Newmann and Penrose (NP), defined by

$$\omega_{AB}\theta^A \wedge \theta^B = \sigma_\alpha Z^\alpha + \bar{\sigma}_\alpha \bar{Z}^\alpha.$$

In the coframe $\{\theta^A\}$, these coefficients may be written as

$$\sigma_\alpha = \sigma_{\alpha A}\theta^A, \qquad \bar{\sigma}_\alpha = \bar{\sigma}_{\alpha A}\bar{\theta}^A, \tag{1.3}$$

where $A \in \{1, 2, 3, 4\}, \alpha \in \{1, 2, 3\}$, and in the same way, the curvature 2-forms Σ_α are defined by

$$\Omega_{AB}\theta^A \wedge \theta^B = \Sigma_\alpha Z^\alpha + \bar{\Sigma}_\alpha \bar{Z}^\alpha.$$

In terms of $\sigma_\alpha, \bar{\sigma}_\alpha$ the covariant derivatives of h_A are expressed by

$$\begin{aligned}
\nabla h_1 &= -\tfrac{1}{4}(\bar{\sigma}_3 + \sigma_3) \otimes h_1 + \tfrac{1}{2}\bar{\sigma}_2 \otimes h_2 + \tfrac{1}{2}\sigma_2 \otimes h_3,\\
\nabla h_2 &= -\tfrac{1}{2}\bar{\sigma}_1 \otimes h_1 + \tfrac{1}{4}(\bar{\sigma}_3 - \sigma_3) \otimes h_2 + \tfrac{1}{2}\sigma_2 \otimes h_4,\\
\nabla h_3 &= -\tfrac{1}{2}\sigma_1 \otimes h_1 - \tfrac{1}{4}(\bar{\sigma}_3 - \sigma_3) \otimes h_3 + \tfrac{1}{2}\bar{\sigma}_2 \otimes h_4,\\
\nabla h_4 &= -\tfrac{1}{2}\sigma_1 \otimes h_2 - \tfrac{1}{2}\bar{\sigma}_1 \otimes h_3 + \tfrac{1}{4}(\sigma_3 + \bar{\sigma}_3) \otimes h_4
\end{aligned} \tag{1.4}$$

(∇ is torsion-less), and the first group of structure equations is given by Israel [4],

$$\begin{aligned}
\mathrm{d}\theta^1 &= -\tfrac{1}{4}(\bar{\sigma}_3 + \sigma_3) \wedge \theta^1 + \tfrac{1}{2}\bar{\sigma}_1 \wedge \theta^2 + \tfrac{1}{2}\sigma_1 \wedge \theta^3,\\
\mathrm{d}\theta^2 &= -\tfrac{1}{2}\bar{\sigma}_2 \wedge \theta^1 + \tfrac{1}{4}(\sigma_3 - \bar{\sigma}_3) \wedge \theta^2 + \tfrac{1}{2}\sigma_1 \wedge \theta^4,\\
\mathrm{d}\theta^3 &= -\tfrac{1}{2}\sigma_2 \wedge \theta^1 + \tfrac{1}{4}(\bar{\sigma}_3 - \sigma_3) \wedge \theta^3 + \tfrac{1}{2}\bar{\sigma}_2 \wedge \theta^4,\\
\mathrm{d}\theta^4 &= -\tfrac{1}{2}\sigma_2 \wedge \theta^2 - \tfrac{1}{2}\bar{\sigma}_2 \wedge \theta^3 - \tfrac{1}{4}(\sigma_3 + \bar{\sigma}_3) \wedge \theta^4.
\end{aligned} \tag{1.5}$$

In consequence of the above, Cartan's first structure equations in $\mathbb{C}^3$ take the form

$$\begin{aligned} dZ^1 &= \tfrac{1}{2}\sigma_3 \wedge Z^1 - \sigma_2 \wedge Z^3, \\ dZ^2 &= \tfrac{1}{2}\sigma_3 \wedge Z^2 + \sigma_1 \wedge Z^3, \\ dZ^3 &= \tfrac{1}{2}\sigma_1 \wedge Z^1 - \tfrac{1}{2}\sigma_2 \wedge Z^2 \end{aligned} \tag{1.6}$$

and similarly for $\bar{Z}^\alpha$. The basis $\{Z^\alpha, \bar{Z}^\alpha\}$ is the 2-form basis in the complex space $\mathbb{C}^3$. On the other hand, Cartan's structure equations involving the curvature forms Σ_α follow immediately from (1.6):

$$d\sigma_1 = \Sigma_1 + \tfrac{1}{2}\sigma_3 \wedge \sigma_1, \qquad d\sigma_2 = \Sigma_2 + \tfrac{1}{2}\sigma_2 \wedge \sigma_3, \qquad d\sigma_3 = \Sigma_3 + \tfrac{1}{2}\sigma_2 \wedge \sigma_1. \tag{1.7}$$

Finally, with respect to the basis $\{Z^\alpha, \bar{Z}^\alpha\}$ of $\mathbb{C}^3$, the curvature 2-forms Σ_α may be expressed as

$$\Sigma_\alpha = (C_{\alpha\beta} - \tfrac{1}{2}K\gamma_{\alpha\beta})Z^\beta + E_{\alpha\bar{\beta}}\bar{Z}^\beta. \tag{1.8}$$

Here the coefficients $C_{\alpha\beta}$ and $E_{\alpha\bar{\beta}}$ denote the components of Weyl's conformal tensor field and the components of the electric tensor field E, respectively [4]. In addition, K and $\gamma_{\alpha\beta}$ are the scalar curvature of (M, g) and the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{1.9}$$

We also recall that $\flat : TM \to T^*M$, $\sharp : T^*M \to TM$ mean the musical isomorphisms defined by g. If Ω is an almost symplectic form, then

$$\Omega^\flat : TM \to T^*M, \qquad Z \mapsto -i_Z\Omega =^\flat Z,$$

denotes the symplectic isomorphism.

2. Hyperbolic skew symmetric Killing vector fields

If (M, g) is a general space–time, then in terms of a Sachs frame $\{h_A\}$, the soldering form dp (or the canonical vector-valued 1-form) is expressed by

$$dp = \theta^A \otimes h_A \Rightarrow g = 2(\theta^1 \otimes \theta^4 - \theta^2 \otimes \theta^3). \tag{2.1}$$

In these conditions, a hyperbolic vector field X on M may be written as

$$X = X^1 h_1 + X^4 h_4, \quad X^1, X^4 \in C^\infty M. \tag{2.2}$$

In the present paper, we assume that any X is a skew symmetric Killing (SSK) vector field having a spatial vector field $Y = Y^2 h_2 + Y^3 h_3$ as generative [8], i.e.

$$\nabla X = X \wedge Y = Y^\flat \otimes X - X^\flat \otimes Y \tag{2.3}$$

($\wedge$: wedge product of vector fields). Taking the covariant differential of X, one finds by (1.4) and (2.2)

$$\nabla X = (\mathrm{d}X^1 - \tfrac{1}{4}X^1(\sigma_3+\bar{\sigma}_3))\otimes h_1 + (\mathrm{d}X^4 + \tfrac{1}{4}X^4(\sigma_3+\bar{\sigma}_3))\otimes h_4 \\ -\tfrac{1}{2}(X^1\bar{\sigma}_2 - X^4\sigma_1)\otimes h_2 + \tfrac{1}{2}(X^1\sigma_2 - X^4\bar{\sigma}_1)\otimes h_3 \tag{2.4}$$

and remembering that θ^1 and θ^4 are the dual forms of h_4 and h_1, respectively, one may write by (2.3)

$$\nabla X = Y^\flat \otimes (X^1 h_1 + X^4 h_4) - (X^1\theta^4 + X^4\theta^1)\otimes(Y^2 h_2 + Y^3 h_3), \tag{2.5}$$

where

$$Y^\flat = -Y^3\theta^2 - Y^2\theta^3 \tag{2.6}$$

is the dual form of Y. By (2.4) and (2.5), a standard calculation gives

$$\mathrm{d}X^1 - \tfrac{1}{4}X^1(\sigma_3+\bar{\sigma}_3) = X^1 Y^\flat, \qquad \mathrm{d}X^4 + \tfrac{1}{4}X^4(\sigma_3+\bar{\sigma}_3) = X^4 Y^\flat, \\ X^1\bar{\sigma}_2 - X^4\sigma_1 = -2Y^2X^\flat, \qquad X^1\sigma_2 - X^4\bar{\sigma}_1 = -2Y^3X^\flat \tag{2.7}$$

and by the first two equations of (2.7) it follows at once that

$$Y^\flat = \frac{\mathrm{d}\|X\|^2}{\|X\|^2} = \frac{\mathrm{d}E}{E}, \tag{2.8}$$

where $\frac{1}{2}\|X\|^2 = E$ is the energy function corresponding to X. Hence one may affirm that the dual form of the generative Y is exact (or Y is a gradient vector field). Next, since

$$X^\flat = X^1\theta^4 + X^4\theta^1, \tag{2.9}$$

one derives by (1.5), (2.6) and (2.7)

$$\mathrm{d}X^\flat = 2Y^\flat \wedge X^\flat \tag{2.10}$$

and so one refinds Rosca's lemma regarding SSK vector fields [8]. One has to notice that in the case under discussion, the recurrence form $2Y^\flat$ is exact. On the other hand, since by hypothesis the last two equations of (2.7) hold good for any hyperbolic vector field, one gets at once

$$\sigma_1 = 2Y^2\theta^1, \quad \bar{\sigma}_1 = 2Y^3\theta^1, \qquad \sigma_2 = -2Y^3\theta^4, \quad \bar{\sigma}_2 = -2Y^2\theta^4. \tag{2.11}$$

The above equations reveal some significant properties. First of all, performing the covariant differential of the generative vector field

$$Y = Y^2 h_2 + Y^3 h_3 \tag{2.12}$$

and taking account that Y is a gradient vector field, one may take

$$\mathrm{d}Y^2 + \tfrac{1}{4}(\bar{\sigma}_3 - \sigma_3)Y^2 = -Y^2Y^3\theta^2, \qquad \mathrm{d}Y^3 - \tfrac{1}{4}(\bar{\sigma}_3 - \sigma_3)Y^3 = -Y^2Y^3\theta^3 \tag{2.13}$$

and one derives on behalf of (2.1),

$$\nabla Y = -Y^2Y^3\,\mathrm{d}p = \tfrac{1}{2}\|Y\|^2\,\mathrm{d}p. \tag{2.14}$$

Consequently, one may affirm that the generative vector field Y of the SSK vector field X is a concurrent vector field [11] having, up to $\frac{1}{2}$, its length as conformal scalar (as is well-known, any concurrent vector field is a gradient). On the other hand, the six 1-forms $\sigma_\alpha, \bar{\sigma}_\alpha$ associated to the CVF may be expressed as

$$\sigma_\alpha = \sigma_{\alpha A}\theta^A, \qquad \bar{\sigma}_\alpha = \bar{\sigma}_{\alpha A}\bar{\theta}^A \tag{2.15}$$

(the "bar" denoting the complex conjugate, i.e. $\theta^2 = \bar{\theta}^3, \theta^1 = \bar{\theta}^1, \theta^4 = \bar{\theta}^4$), where the coefficients $\sigma_{\alpha A}, \bar{\sigma}_{\alpha A}$ correspond to the 12 spinorial coefficients of NP [4]. From (2.11) one gets at once

$$\sigma_{13} = 0, \qquad \sigma_{14} = 0, \qquad \sigma_{21} = 0, \qquad \sigma_{22} = 0, \tag{2.16}$$

which in terms of CVF characterize a space–time of type D in Petrov's classification. Since $\|Y\|^2 = -2Y^2Y^3$, one derives by (2.6), (2.8) and (2.13)

$$\frac{\mathrm{d}\|Y\|^2}{\|Y\|^2} = Y^\flat = \frac{\mathrm{d}\|X\|^2}{\|X\|^2} \Rightarrow \|X\|^2 = c\|Y\|^2, \quad c = \text{constant} \tag{2.17}$$

(i.e. the energy functions of X and Y are homothetic).

In another order of ideas, setting

$$2f = -2Y^2Y^3 = \|Y\|^2, \tag{2.18}$$

one has by (2.19)

$$\nabla f = fY \Rightarrow \|\nabla f\|^2 = -2f^3 \tag{2.19}$$

and since one finds

$$\operatorname{div} Y = 2f^2, \tag{2.20}$$

one derives

$$\operatorname{div}(\nabla f) = -2f^3. \tag{2.21}$$

Hence, since the function $f : \mathbf{R}^4 \to \mathbf{R}$ has the property that both $\|\nabla f\|^2$ and div (∇f) are functions of f, it follows that f is an *isoparametric* function (i.e. the energy functions of X and Y are isoparametric functions). On the other hand, by (2.5) and (2.14), one derives by a standard calculation

$$[X, Y] = \|Y\|^2 X.$$

Hence, by a known definition, the vector field X admits an infinitesimal conformal transformation of generator Y. Further, operating on (2.3) by the exterior covariant derivative operator d^∇, one obtains by (2.3), (2.10) and (2.14)

$$\mathrm{d}^\nabla(\nabla X) = \nabla^2 X = \|Y\|^2 X^\flat \wedge \mathrm{d}p + (X^\flat \wedge Y^\flat) \otimes Y. \tag{2.22}$$

This affirms that X is a quasi-exterior concurrent vector field with respect to Y [8] (see also [6]). Next, if R denotes the curvature tensor field, then by the general formula

$$R(Z, Z')W = \nabla^2 W(Z, Z'), \tag{2.23}$$

one infers

$$R(X, Y)X = \tfrac{1}{2}\|Y\|^4\|X\|^2. \tag{2.24}$$

In another order of ideas, let

$$\varphi_{\rm h} = \theta^1 \wedge \theta^4, \tag{2.25}$$

$$\varphi_{\rm s} = \theta^2 \wedge \theta^3 \tag{2.26}$$

be the simple unit forms corresponding to the hyperbolic distribution $D_{\rm h} = \{h_1, h_4\}$ and the spatial distribution $D_{\rm s} = \{h_2, h_3\}$. By the structure equations of (1.5) and with the help of (2.11), one gets

$$\mathrm{d}\varphi_{\rm h} = 2Y^\flat \wedge \varphi_{\rm h}, \tag{2.27}$$

$$\mathrm{d}\varphi_{\rm s} = 0. \tag{2.28}$$

Hence, if $Z_{\rm s} \in D_{\rm s}$, $Z_{\rm h} \in D_{\rm h}$, one derives from (2.27) and (2.28)

$$\mathcal{L}_{Z_{\rm s}}\varphi_{\rm h} = Y^\flat(Z_{\rm s})\varphi_{\rm h}, \qquad \mathcal{L}_{Z_{\rm h}}\varphi_{\rm s} = 0,$$

which, following a known definition, shows that $\varphi_{\rm h}$ is a *conformal integral* invariant of $D_{\rm s}$ and $\varphi_{\rm s}$ is an *integral* invariant of $D_{\rm h}$.

Therefore, by Frobenius' theorem, one may affirm that the manifold M under consideration is foliated by surfaces $M_{\rm s}$ and $M_{\rm h}$ tangent to $D_{\rm s}$ and $D_{\rm h}$, respectively. One also finds that on $M_{\rm h}$ (resp. $M_{\rm s}$), one has

$$\langle\nabla h_2, \nabla h_2\rangle = 0, \qquad \langle\nabla h_3, \nabla h_3\rangle = 0, \qquad \langle\nabla h_1, \nabla h_1\rangle = 0, \qquad \langle\nabla h_4, \nabla h_4\rangle = 0$$

and consequently, by referring to [7], one may say that $M_{\rm h}$ and $M_{\rm s}$ are *totally pseudo-isotropic* surfaces of M. Moreover, the spatial surface $M_{\rm s}$ is totally geodesic, i.e.

$$\langle \mathrm{d}p_{\rm s}, \nabla h_1\rangle = 0, \qquad \langle \mathrm{d}p_{\rm s}, \nabla h_4\rangle = 0,$$

where $\mathrm{d}p_{\rm s}$ is the soldering form of the surface $M_{\rm s}$.

Summarizing, we state the following.

Theorem 1. *Let M be a general space–time carrying a totally hyperbolic SSK vector field X and let Y be its spatial generative. Then any such manifold M is the local Riemannian product*

$$M = M_{\rm h} \times M_{\rm s},$$

where $M_{\rm h}$ is a hyperbolic surface and $M_{\rm s}$ a spatial surface, which are such that the immersions

$$x_{\rm h} : M_{\rm h} \to M, \qquad x_{\rm s} : M_{\rm s} \to M$$

are totally pseudo-isotropic and $x_{\rm s} : M_{\rm s} \to M$ is totally geodesic.

In addition

1. *any such M is a space–time of type D in Petrov's classification*;
2. *the square $\|X\|^2$ and $\|Y\|^2$ of X and Y are isoparametric functions*;

3. *Y defines an infinitesimal conformal transformation of X*;
4. *X is a quasi-exterior concurrent vector field*;
5. *the curvature tensor field R satisfies*

$$R(X, Y)X = \frac{1}{2}\|Y\|^4\|X\|^2.$$

3. Second order properties

In this section, some second order properties are discussed and the congruence of Debever $\Gamma(h_4)$ is studied. Regarding the second order properties involving the forms Z^α which defines the complex $\mathbb{C}^3$-basis, a series of properties also appear. In terms of CVF, the transcription of the second of Cartan's structure equations (see (1.7)) regarding the curvature forms Σ_α are

$$\mathrm{d}\sigma_1 = \Sigma_1 + \tfrac{1}{2}\sigma_3 \wedge \sigma_1, \qquad \mathrm{d}\sigma_2 = \Sigma_2 + \tfrac{1}{2}\sigma_2 \wedge \sigma_3, \qquad \mathrm{d}\sigma_3 = \Sigma_3 + \tfrac{1}{2}\sigma_2 \wedge \sigma_1. \quad (3.1)$$

Hence, in the case under discussion, one infers by a standard calculation by (1.5), (2.11) and (2.13)

$$\mathrm{d}\sigma_1 = Y^2(\sigma_3 - 2Y^3\theta^2 + 2Y^\flat) \wedge \theta^1, \qquad \mathrm{d}\sigma_2 = Y^3(\sigma_3 + 2Y^2\theta^3 - 2Y^\flat) \wedge \theta^4,$$
$$\mathrm{d}\sigma_3 = -4\|Y\|^2\theta^2 \wedge \theta^3 \quad (3.2)$$

and consequently Eq. (3.1) moves to

$$\Sigma_1 = -2Y^2(Y^3\theta^2 - Y^\flat) \wedge \theta^1, \qquad \Sigma_2 = 2Y^3(Y^2\theta^3 - Y^\flat) \wedge \theta^4,$$
$$\Sigma_3 = \theta^2 \wedge \theta^3 - \theta^1 \wedge \theta^4. \quad (3.3)$$

In terms of the basis $\{Z^\alpha, \bar{Z}^\alpha\}$ of $\mathbb{C}^3$, one may write (see also [4])

$$Z^1 = \theta^3 \wedge \theta^4, \quad Z^2 = \theta^1 \wedge \theta^2, \quad Z^3 = \tfrac{1}{2}(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3),$$
$$\bar{Z}^1 = \theta^2 \wedge \theta^4, \quad \bar{Z}^2 = \theta^1 \wedge \theta^3, \quad \bar{Z}^3 = \tfrac{1}{2}(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3) \quad (3.4)$$

and Eq. (3.3) turns out to

$$\Sigma_1 = -2\|Y\|^2 Z^2 + 2(Y^2)^2\bar{Z}^2, \qquad \Sigma_2 = -2\|Y\|^2 Z^1 + 2(Y^3)^2\bar{Z}^1,$$
$$\Sigma_3 = -2Z^3. \quad (3.5)$$

We recall that the null vector field h_4, which is called Debever's vector field [4], plays a distinguished role in the frame of the CVF. Since by (2.11) one has

$$\sigma_{14} = 0, \qquad \sigma_{13} = 0 \Leftrightarrow \sigma_1 \wedge Z^2 = 0, \quad (3.6)$$

then by [4] the congruence $\Gamma(h_4)$ is said to be geodesic and shear 1-free. On the other hand, with respect to the basis $\{Z^\alpha, \bar{Z}^\alpha\}$ of $\mathbb{C}^3$, the curvature 2-forms may be expressed by (1.8) (see [1]). By (3.5) one finds

$$\Sigma_1 = (C_{12} - \tfrac{1}{2}K)Z^2 + E_{1\bar{2}}\bar{Z}^2 \quad (3.7)$$

and since one gets

$$C_{12} = \bar{\sigma}_{12}, \tag{3.8}$$

then in terms of the CVF, the above equation proves that the congruence of Debever $\Gamma(h_4)$ is of electric type [4]. We mention that we have

$$C_{\alpha\beta} = \begin{pmatrix} 0 & C_{12} & 0 \\ C_{12} & 0 & 0 \\ 0 & 0 & 4C_{12} \end{pmatrix} \tag{3.9}$$

and this agrees with the fact that the manifold under consideration is of type D in the sense of Petrov [1,4].

Perform now the differentials of the 2-forms $\{Z^\alpha, \bar{Z}^\alpha\}$, which define the complex $\mathbb{C}^3$-basis. Then, by (1.5), (2.11) and (2.13) and taking account of (2.6), one infers

$$\begin{aligned} &\mathrm{d}Z^1 = \tfrac{1}{2}\bar{\sigma}_3 \wedge Z^1, \quad \mathrm{d}Z^2 = \tfrac{1}{2}\bar{\sigma}_3 \wedge Z^2, \quad \mathrm{d}Z^3 = 2Y^\flat \wedge Z^3, \\ &\mathrm{d}\bar{Z}^1 = \tfrac{1}{2}\sigma_3 \wedge \bar{Z}^1, \quad \mathrm{d}\bar{Z}^2 = \tfrac{1}{2}\sigma_3 \wedge \bar{Z}^2, \quad \mathrm{d}\bar{Z}^3 = 2Y^\flat \wedge \bar{Z}^3. \end{aligned} \tag{3.10}$$

Therefore, from the above, one may affirm that the space–time under consideration is endowed with an exterior recurrent complex basis. Associated with the $\mathbb{C}^3$-basis, one may consider the almost symplectic forms

$$\Omega_i = \lambda_i(\bar{Z}^3 + Z^3) + C_i(\bar{Z}^3 - Z^3), \tag{3.11}$$

$\lambda_i \in C^\infty M,\ C_i = \text{constant}, i = 1, 2$. By (1.6), (2.11) and (2.13), one finds that the necessary and sufficient condition in order that the pairing Ω_i be symplectic forms, i.e. $\mathrm{d}\Omega_i = 0$, is expressed by the conditions

$$\mathrm{d}\lambda_i + 2\lambda_i Y^\flat = 0. \tag{3.12}$$

Next, by referring to [3], we agree to say that (Ω_1, Ω_2) defines a *nearly symplectic couple* if $\Omega_1 \wedge \Omega_2 = 0$. Hence, the scalars λ_i and the constants C_i are related by

$$\lambda_1 C_2 + \lambda_2 C_1 = 0. \tag{3.13}$$

In consequence of (2.8), one may write

$$Y^\flat = \frac{\mathrm{d}\|X\|^2}{\|X\|^2}$$

and since ${}^\flat Y = Y^2\theta^2 - Y^3\theta^3$ defines the symplectic isomorphism to Y, one derives by (1.6) and (2.13)

$$\mathrm{d}^\flat Y = \mathcal{L}_Y \Omega_i = 0.$$

This affirms that the symplectic forms Ω_i are invariant by the generative vector field Y of X.

Thus, we may state the following.

Theorem 2. *Any space–time (M, g) which carries a total hyperbolic SSK vector field X having a spatial vector field Y as generative is structured by an exterior recurrent complex $\mathbb{C}^3$-basis.*

The congruence of Debever $\Gamma(h_4)$ associated with M is of electric type.

Further, let Ω_i $(i = 1, 2)$ be the almost symplectic forms associated with the $\mathbb{C}^3$-basis and let λ_i be the scalars associated with Ω_i.

Then the necessary and sufficient condition in order that Ω_i be symplectic is that λ_i be conformal to $\|X\|^2$, and in this case Ω_i are invariant by the generative Y and they define a nearly symplectic couple in the sense of [3].

For further reading see [2,5,9,10].

References

[1] M. Cahen, R. Debever, L. Defrise, A complex vectorial formalism in general relativity, J. Math. Mech. 16 (1967) 749–785.

[2] E. Cartan, Systemes Différentiels Extérieurs et Leurs Applications Géométriques, Hermann, Paris, 1945.

[3] H. Geiges, Symplectic couples on 4-manifolds, Duke Math. J. 85 (1996) 701–711.

[4] W. Israel, Differential Forms in General Relativity, Dublin Inst. Advanced Studies, 1970.

[5] I. Mihai, R. Rosca, L. Verstraelen, Some Aspects of the Differential Geometry of Vector Fields, Vol. 2, PADGE, KU Leuven, KU Brussel, 1996.

[6] M. Petrovic, R. Rosca, L. Verstraelen, Exterior concurrent vector fields on a Riemannian manifold, Soochow J. Math. 15 (1989) 179–187.

[7] R. Rosca, Les variétés pseudo-isotropes dans un espace-temps de Minkowski, C.R. Acad. Sci. Paris 270 (1970) 1071–1073.

[8] R. Rosca, On exterior concurrent skew-symmetric Killing vector fields, Rend. Sem. Mat. Messina 2 (1993) 131–145.

[9] G. Vrănceanu, R. Rosca, Introduction in Relativity and Pseudo-Riemannian Geometry, Edit. Acad. Române, Bucharest, 1976.

[10] A. West, Isoparametric systems, in: Geometry and Topology of Submanifolds, World Scientific, River Edge, 1989, pp. 222–230.

[11] K. Yano, B.Y. Chen, On the concurrent vector fields of immersed manifolds, Kodai Math. Sem. Rep. 23 (1971) 343–350.